Off-shell Colour–Kinematics Duality and the BRST-Lagrangian Double Copy

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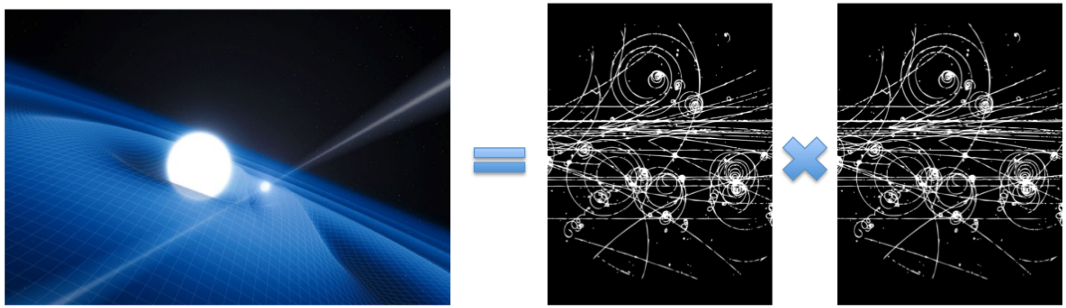
Based on joint work 2007.13803, 2102.11390, 2108.03030 and 22xx.xxxxx with Branislav Jurčo, Hyungrok Kim, Tommaso Macrelli, Christian Saemann and Martin Wolf (BJKMSW)

Gravity $=$ Gauge $\times$ Gauge



$g_{\mu\nu}$ $\qquad$ $A_{\mu}{}^{a}$ $\qquad$ $A_{\nu}{}^{b}$

- Is gravity the double copy of the other fundamental forces of Nature?
- Long history and many guises **[Feynman; Papini; Kawai, Lewellen, Tye; Berends, Giele, Kuijf; Bern, Dixon, Dunbar, Perelstein , Rozowsky. . .]**

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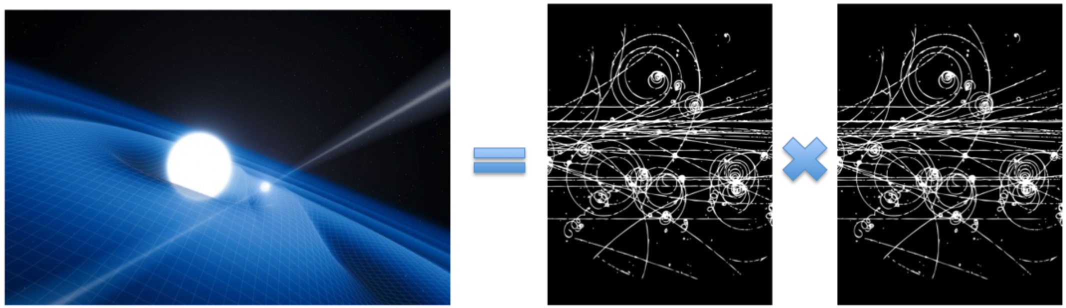

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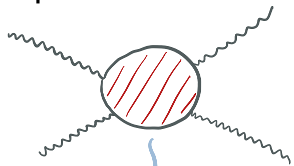
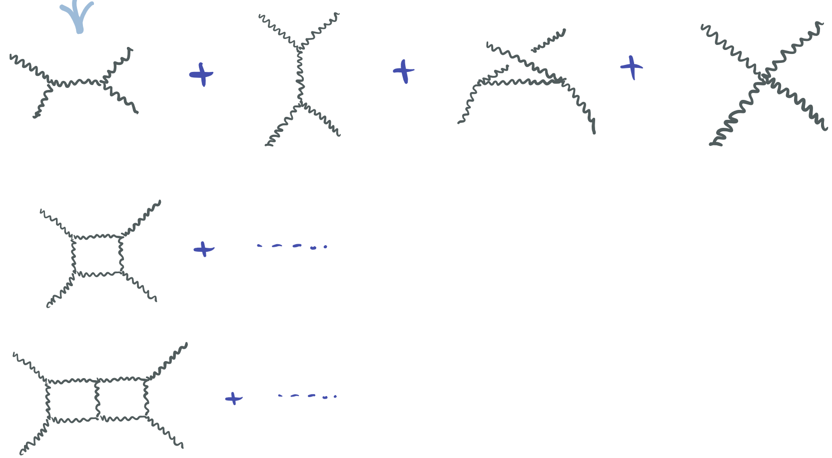
- Renaissance: Bern–Carrasco–Johansson Colour–Kinematics (CK) duality conjecture and double copy of gauge theory and gravity scattering amplitudes [Bern, Carrasco, Johansson '08, '10; Bern, Dennen, Huang, Kiermaier '10]

Scattering Amplitudes

$\rightarrow$ Physical observables tested at particle accelerators (e.g. Large Hadron Collider)



Feynman diagram expansion



Explosion of complexity

Hidden simplicity:
On-shell amplitudes paradigm

Deep, hidden
structures

$\rightarrow$ New insights into the underlying theories themselves

Colour–Kinematics Duality

- Amplitude for gluons to scatter schematically:

Colour numerators $c \sim f_{ab}{}^{c} f_{cd}{}^{e} \cdots$
colour/gauge group data of gluons

Kinematic numerators $n \sim \varepsilon_{\mu} p^{\mu} \cdots$
polarisation and momentum data of gluons

$$\mathcal{A}^{n,L}_{\text{gluons}} \sim \sum_{i} \int_{\text{loops}} \frac{c_i n_i}{d_i}$$

Sum over cubic diagrams

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Sum over cubic diagrams

Kinematic Jacobi identity

- Bern-Carrasco-Johansson CK duality conjecture 2008:

Jacobi identity

$$c_i + c_j + c_k = 0 \qquad \Rightarrow \qquad n_i + n_j + n_k = 0$$

- Proven at tree level [Stieberger '09; Bjerrum, Bohr, Damgaard, Vanhove '09; Du, Teng '16; Bridges, Mafra '19; Mizera '19; Reiteer '19. . .]

- Conjectured at loop level with highly non-trivial examples [Bern, Carrasco, Johansson '08 '10; Carrasco, Johansson '11; Bern, Davies, Dennen, Huang, Nohle '13; Bern, Davies, Dennen '14. . .]

The Double Copy Prescription

- Assuming CK duality is realised, gravity comes for free:
  [Bern, Carrasco, Johansson '08, '10; Bern, Dennen, Huang, Kiermaier '10]

$$\mathcal{A}^{n,L}_{\text{gluons}} \sim \sum_i \int_{\text{loops}} \frac{c_i n_i}{d_i}$$

Double copy kinematics

$$c_i \longrightarrow n_i$$

$$\mathcal{A}^{n,L}_{\text{gravitons}} \sim \sum_i \int_{\text{loops}} \frac{n_i n_i}{d_i}$$

- 'Gluons for (almost) nothing, gravitons for free' JJ Carrasco

Generalisations, Implications and Applications

- Growing zoology of generalisations: ϕ^3 theory, Maxwell/scalar/Yang-Mills supergravity, gauged supergravity (Minkowski vacua), non-linear sigma model, pure gravity, Born-Infeld, conformal gravity, Z-theory, Navier-Stokes fluids, topologically massive Yang–Mills, geometric/world-sheet and pure spinor formalisms, ambitwistor string theories, scattering equations, non-trivial gluon and spacetime backgrounds, special Galileons, massive gravity, EFT. . . [Hodges '11; Broedel, Carrasco '11; Bern, Boucher–Veronneau, Johansson '11; Broedel, Dixon '12; Bargheer, He,McLoughlin '12; Huang, Johansson '12; Cachazo, He, Yuan '13 '14; Dolan, Goddard '13; Naculich '14 '15; Mason, Skinner '13; Adamo, Casali, Skinner '13; Adamo, Casali, Mason, Nekovar '17 '18; Geyer, Monteiro '18; LB '18; Geyer, Mason '19; LB-Duff-Marrani '19; Geyer, Monteiro, Stark, Muchão '21; Chiodaroli, Günaydin, Johansson, Roiban '14 '15; Johansson, Ochirov '15 '16; Chiodaroli, Günaydin, Johansson, Roiban '17; Carrasco, Mafra, Schlotterer '16; Johansson, Nohle '17; Azevedo, Chiodaroli, Johansson, Schlotterer '18; Farrow, Lipstein, McFadden '19; Momeni, Rumbutis, Tolley '20; Johnson, Jones, Paranjape '20; Zhou '21; Diwakar, Herderschee, Roiban, Teng '21; González, Momeni, Rumbutis '21; Cheung, Parra-Martínez, Sivaramakrishnan '22, González, Liang, Mark Trodden '22. . .]
- Uncovering new facets of perturbative quantum gravity: miraculous cancelations, anomalies, unchartered amplitudes . . . [Carrasco, Johansson '11; Bern, Davies, Dennen, Huang, Nohle '13; Bern, Davies, Dennen '14; Bern, Carrasco, Chen, Edison, Johansson, Parra-Martínez, Roiban, Zeng '18; Bern, Kosower, Parra, Martínez '20. . .]
- Applications to classical general relativity: (non)perturbative solutions, black holes merger modelling [Monteiro, O'Connell, White '14; Cardoso, Nagy, Nampuri '16; Luna, Monteiro, Nicholson, Ochirov, O'Connell, Westerberg, White '16; Berman, Chacón, Luna, White '18; Kosower, Maybee, O'Connell '18; Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Bern, Luna, Roiban, Shen, Zeng '20; Chacón-Nagy, White '21; Adamo, Cristofoli, Ilderton '22. . .]

Elucidating Colour–Kinematics Duality and the Double Copy

Manifold perspectives (hopelessly partial list)

- Structural properties, e.g. geometric, graph theoretic, analytic, algebraic [Carrasco, Johansson '11; Broedel, Carrasco '11; de la Cruz, Kniss, Weinzierl '17; Mizera '19; Reiterer '19. . .]
- String theory and pure spinors [Stieberger '09; Bjerrum, Bohr, Damgaard, Vanhove '09; Mafra, Schlotterer, Stieberger '11; Broedel, Schlotterer, Stieberger '13; Mafra, Schlotterer '14 '15; Carrasco, Mafra, Schlotterer '16; Casali, Mizera, Tourkine '20; Bridges, Mafra '21. . .]
- Kinematic Algebras [Monteiro, O'Connell '11, '13; Bjerrum, Bohr, Damgaard, Monteiro, O'Connell '12; Fu, Krasnov '16; Chen, Johansson, Teng, Wang 19; Campiglia, Nagy '21; Frost, Mason '20; Brandhuber, Chen, Johansson, Travaglini, Wen '21. . .]
- Classical double copy [Monteiro, O'Connell, White '14; Luna, Monteiro, O'Connell, White '15; Cardoso, Nagy, Nampuri '16; Berman, Chacón, Luna, White '18; Bahjat, Abbas, Stark, Muchão, White '20; White '20; Chacón-Nagy-White '21, Emond, Moynihan '22. . .]
- Ambitwistors and scattering equations [Cachazo, He, Yuan '13 '14; Mason, Skinner '13; Adamo, Casali, Skinner '13; Adamo, Casali, Mason, Nekovar '17 '18; Geyer, Monteiro '18; Geyer, Mason '19; Geyer, Monteiro, Stark, Muchão '21. . .]
- KLT bootstrap [Chi, Elvang, Herderschee, Jones, Paranjape '21]
- Covariant CK duality [Cheung, Mangan '21; Moynihan '21. . .]
- Lagrangian CK duality and double copy [Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13; Cheung, Shen '16; LB, Nagy '20; Ben–Shahar, Johansson '21. . .]

The Becchi–Rouet–Stora–Tyutin Complex

- Off-shell symmetry considerations lead one to naturally consider the full BRST or Batalin–Vilkovisky (BV) complex

Antifields

Nakanishi–Lautrup

Ghost

$$
\begin{tikzcd}[column sep=small]
 & & \overset{A^a_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \arrow[rr, "\delta^\mu_\nu\Box-\partial_\nu\partial^\mu"] & & \overset{A^{+a}_\mu}{\Omega^1(\mathbb{M}^d)\otimes\mathfrak{g}} \arrow[ddrr, "-\partial^\mu"] & & \\
 & & \overset{b^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} \arrow[drr, "1"] & & \overset{b^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}} & & \\
\underbrace{\overset{c^a}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_0} \arrow[uurr, "-\partial_\mu"] & & \underbrace{\overset{\bar{c}^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_1} \arrow[urr, "-1"'] & & \underbrace{\overset{\bar{c}^{a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_2} & & \underbrace{\overset{c^{+a}}{\mathscr{C}^\infty(\mathbb{M}^d)\otimes\mathfrak{g}}}_{\mathfrak{L}_3}
\end{tikzcd}
$$

[Anastasiou, LB, Duff, Hughes, Nagy Zoccali '14 '18; BJKMSW '20, '21, '22]

- See also the pure spinor BRST cohomology approach e.g. [Mafra, Schlotterer, Stieberger '11; Mafra, Schlotterer '14 '15; Bridges, Mafra '21]

Logical Conclusion

- CK duality can be realised as an infinite dimensional anomalous symmetry of Yang–Mills BRST action [BJKMSW '20, '21]

Kinematic algebra index
(DeWitt index over all field species
with ∞ tower of auxiliaries)

Colour metric

Kinematic metric

Colour algebra index

$$C_{ij}\, c_{ab} A^{ia} \Box A^{ja} + F_{ijk}\, f_{abc} A^{ia} A^{jb} A^{kc}$$

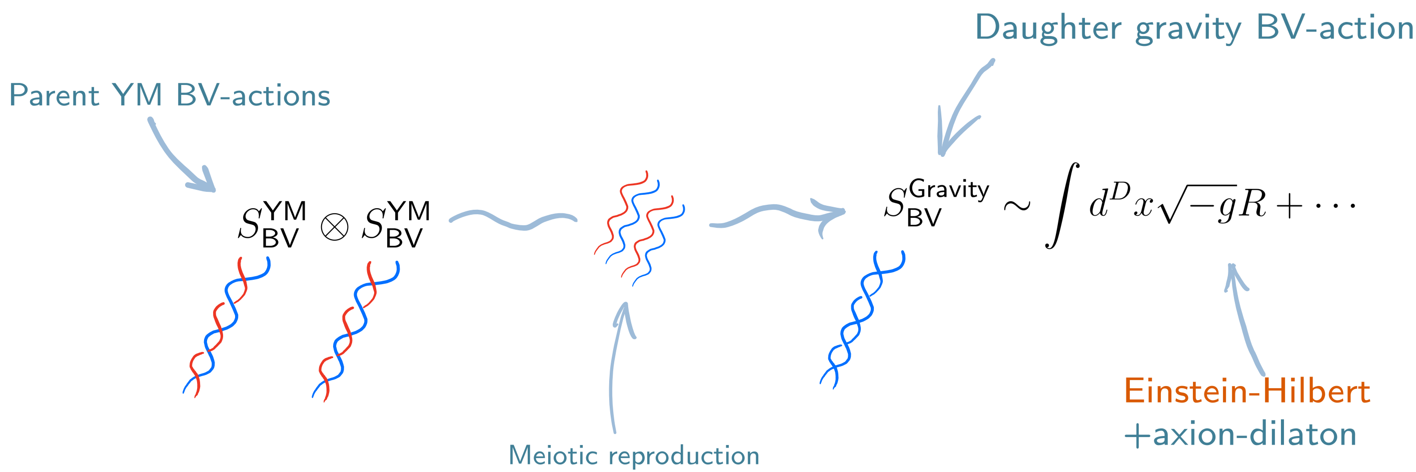
Colour structure constants

Kinematic structure constants

- Natural, but non-standard, notion of CK duality:
  - Natural: symmetry of action
  - Non-standard: loop integrands CK dual, but. . .
  - . . . may be a CK anomaly due to Jacobian counterterms for unitarity
  - Then generalised unitarity proof does not apply, at least not straightforwardly
- However: double copy of BRST action is manifestly valid [BJKMSW '20, '21]

Syngamic Reproduction

- Batalin–Vilkovisky (BV) double copy **[BJKMSW '20; '21, '22 (to appear)]**



(The DNA allusion doesn't actually work, but the picture conveys the idea!)

- Double copy origin of symmetries:

$$\underbrace{(\text{gauge, global susy, R-sym}\ldots)}_{\text{(super) Yang, Mills symmetries}} \longrightarrow \underbrace{(\text{diffeomorphism, local susy, R-sym}\ldots)}_{\text{(super)gravity symmetries}}$$

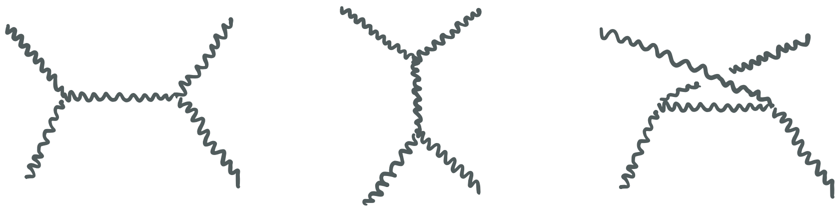
Outline

Phase 1: Colour–Kinematics Duality and Double Copy Review

Amplitudes and Cubic Diagrams

- Can write n-point L-loop gluon amplitude in terms of only cubic diagrams:

$$A_{\mathrm{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$



- c_i: colour numerator, built from f^{abc}, read off diagram i
- n_i: kinematic numerator, built from p, ε ⟵ Non-unique
- d_i: propagator, $\prod_{\text{int. lines}} p^2$, read off diagram i

Amplitudes and Cubic Diagrams

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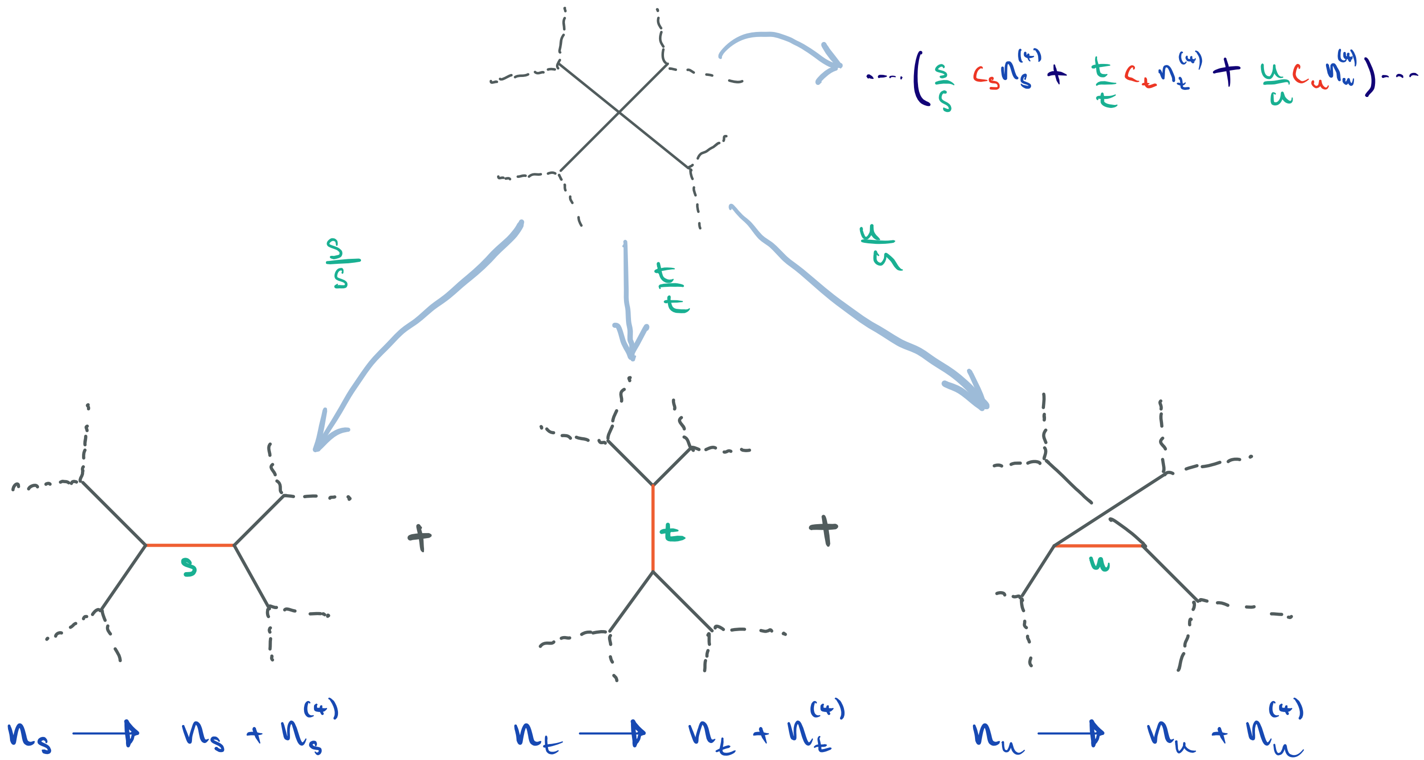


Amplitudes and Cubic Diagrams

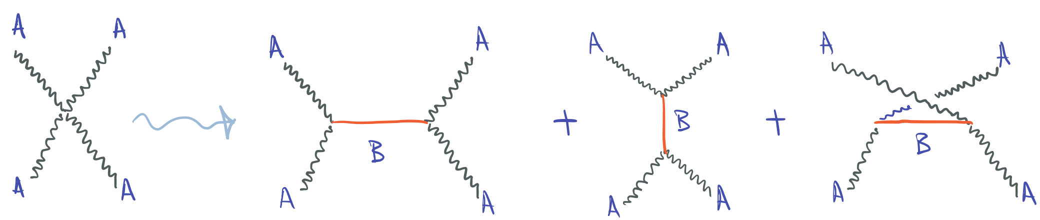
- Can be realised in the Lagrangian through auxiliary fields:
  [Bern, Dennen, Huang, Kiermaier '10]

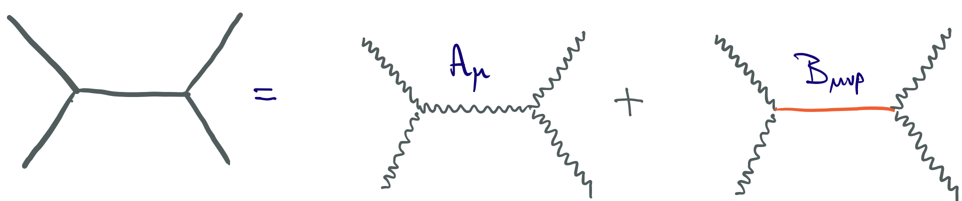
$$g^2[A_\mu, A_\nu][A^\mu, A^\nu] \;\to\; \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \frac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu]$$

- Feynman diagrams give 'cubic' amplitudes directly:

$$A^{n,L}_{\mathrm{YM}} = \sum_{\alpha \in \text{Feynman diag}} \int_L \frac{c_\alpha n_\alpha}{S_\alpha d_\alpha} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

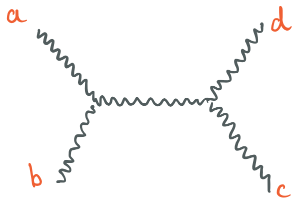
- Example: 4-point s-channel diagram



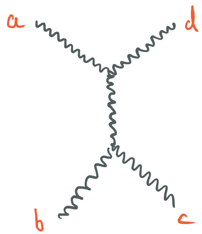


$$n_s = n_s^A + n_s^B$$

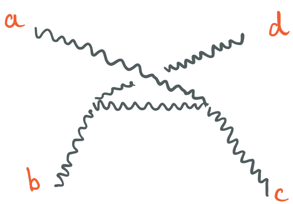
Tree-level 4-point Colour–Kinematics Duality



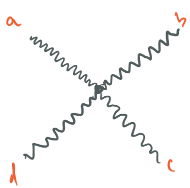
$$= -ig^2 \frac{f^{abe} f_e{}^{cd} n_s}{s} =: -ig^2 \frac{c_s n_s}{s}$$



$$= -ig^2 \frac{f^{aed} f_e{}^{bc} n_t}{t} =: -ig^2 \frac{c_t n_t}{t}$$



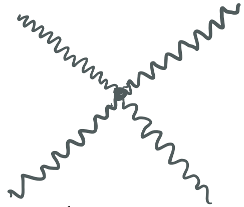
$$= -ig^2 \frac{f^{aec} f_e{}^{db} n_u}{u} =: -ig^2 \frac{c_u n_u}{u}$$



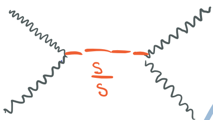
$$= -ig^2 \left(c_s n_s^{(4)} - c_t n_t^{(4)} - c_u n_u^{(4)} \right)$$

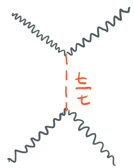
$$n_s = 4\left[(\varepsilon_1 \cdot p_2)\varepsilon_2 - (\varepsilon_2 \cdot p_1)\varepsilon_1 + \tfrac{1}{2}(\varepsilon_1 \cdot \varepsilon_2) p_{12}\right] \cdot \left[(\varepsilon_3 \cdot p_4)\varepsilon_4 - (\varepsilon_4 \cdot p_3)\varepsilon_3 + \tfrac{1}{2}(\varepsilon_3 \cdot \varepsilon_4) p_{34}\right]$$

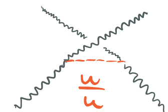
Tree-level 4-point Colour–Kinematics Duality



$$\frac{1}{s} c_s s n_s^{(4)} - \frac{1}{t} c_t t n_t^{(4)} - \frac{1}{u} c_u u n_u^{(4)}$$







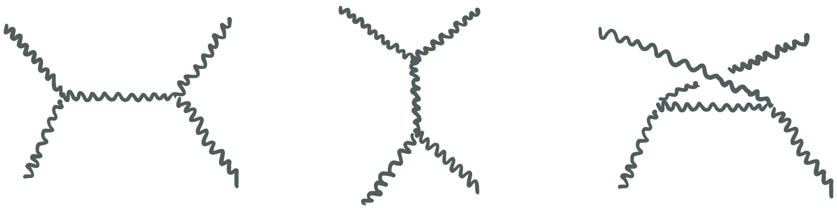
$$\frac{c_s(n_s + s n_s^{(4)})}{s} \qquad \frac{c_t(n_t - t n_t^{(4)})}{t} \qquad \frac{c_u(n_u - u n_u^{(4)})}{u}$$

$$\tilde{n}_s$$

Tree-level 4-point Colour–Kinematics Duality

- Amplitude now sum over purely trivalent diagrams:

$$A_4 \quad \propto \quad \frac{c_s \tilde{n}_s}{s} \quad + \quad \frac{c_t \tilde{n}_t}{t} \quad + \quad \frac{c_u \tilde{n}_u}{u}$$



- Obvious (by Jacobi): $c_s - c_t - c_u = 3f^{ea[b}f_e{}^{cd]} \;=\; 0$

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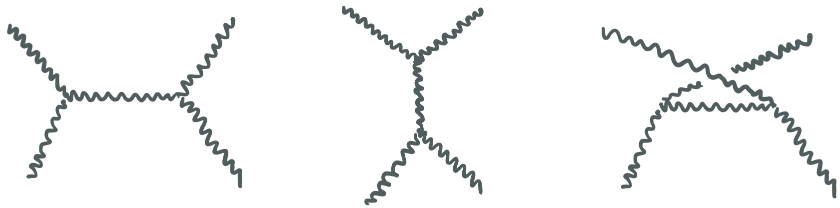

- Obvious (by Jacobi): $c_s - c_t - c_u = 3 f^{ea[b} f_e{}^{cd]} \; = \; 0$

- Not obvious (but true): $\tilde{n}_s - \tilde{n}_t - \tilde{n}_u = 0$
  [Zhu '80]

- Kinematics appears to be playing by the same rules as the colour!

BCJ Colour–Kinematic Duality Conjecture

- Exploiting ambiguity of n_i there is an organisation of the n-point L-loop gluon amplitude:

$$A_{\mathrm{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \int_L \frac{c_i n_i}{S_i d_i}$$

such that

$$\boxed{\begin{array}{ccc} c_i + c_j + c_k = 0 & \Rightarrow & n_i + n_j + n_k = 0 \\ c_i \longrightarrow -c_i & \Rightarrow & n_i \longrightarrow -n_i \end{array}}$$

[Bern, Carrasco, Johansson '08]

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  [Bern, Carrasco, Johansson '08]

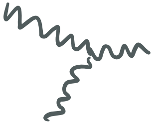
- CK duality established at tree-level
  [Stieberger '09, Bjerrum–Bohr, Damgaard, Vanhove '09... Mizera '19; Reiterer '19]

- Significant evidence up to 4 loops in various (super)YM theories
  [Carrasco, Johansson '11; Bern, Davies, Dennen, Huang, Nohle '13; Bern, Davies, Dennen '14...]

- Quickly becomes difficult to check: remains conjectural for loops (but see later)
  [Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, Zeng '18]

Gravity vs Gauge Feynman Diagrams

- Gluon three point vertex:



$$= gf_{abc}[(p^\rho - q^\rho)\eta^{\mu\nu} + (q^\mu - r^\mu)\eta^{\nu\rho} + (r^\nu - p^\nu)\eta^{\rho\mu}]$$

Gravity vs Gauge Feynman Diagrams

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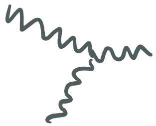

$$= gf_{abc}[(p^\rho - q^\rho)\eta^{\mu\nu} + (q^\mu - r^\mu)\eta^{\nu\rho} + (r^\nu - p^\nu)\eta^{\rho\mu}]$$

- Compare graviton three point [DeWitt '69; Carrasco '15 (TASI lectures)]:

$$
\begin{aligned}
&\frac{\delta S^3}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \to 2\eta^{\mu\tau}\eta^{\nu\sigma}k_1^{\lambda}k_1^{\rho} + 2\eta^{\mu\sigma}\eta^{\nu\tau}k_1^{\lambda}k_1^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_1^{\lambda}k_1^{\rho} + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}k_1^{\sigma}k_1^{\rho} + 2\eta^{\lambda\sigma}\eta^{\mu\nu}k_1^{\tau}k_1^{\rho} + \eta^{\mu\tau}\eta^{\nu\sigma}k_2^{\lambda}k_1^{\rho} + \eta^{\mu\sigma}\eta^{\nu\tau}k_2^{\lambda}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_2^{\mu}k_1^{\rho} + \\
&\eta^{\lambda\sigma}\eta^{\nu\tau}k_2^{\mu}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_2^{\nu}k_1^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_2^{\nu}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_3^{\mu}k_1^{\rho} + \eta^{\lambda\sigma}\eta^{\nu\tau}k_3^{\mu}k_1^{\rho} - \\
&\eta^{\lambda\nu}\eta^{\sigma\tau}k_3^{\mu}k_1^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_3^{\nu}k_1^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_3^{\nu}k_1^{\rho} - \eta^{\lambda\mu}\eta^{\sigma\tau}k_3^{\nu}k_1^{\rho} + \eta^{\lambda\nu}\eta^{\mu\tau}k_3^{\sigma}k_1^{\rho} + \\
&\eta^{\lambda\mu}\eta^{\nu\tau}k_3^{\sigma}k_1^{\rho} + \eta^{\lambda\nu}\eta^{\mu\sigma}k_3^{\tau}k_1^{\rho} + \eta^{\lambda\mu}\eta^{\nu\sigma}k_3^{\tau}k_1^{\rho} + 2\eta^{\mu\nu}\eta^{\rho\tau}k_1^{\lambda}k_1^{\sigma} + 2\eta^{\mu\nu}\eta^{\rho\sigma}k_1^{\lambda}k_1^{\tau} - \\
&2\eta^{\lambda\rho}\eta^{\mu\nu}k_1^{\sigma}k_1^{\tau} + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_1^{\sigma}k_1^{\tau} + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_1^{\sigma}k_1^{\tau} + \eta^{\mu\tau}\eta^{\nu\rho}k_1^{\sigma}k_2^{\lambda} + \eta^{\mu\rho}\eta^{\nu\tau}k_1^{\sigma}k_2^{\lambda} + \\
&\eta^{\mu\sigma}\eta^{\nu\rho}k_1^{\tau}k_2^{\lambda} + \eta^{\mu\rho}\eta^{\nu\sigma}k_1^{\tau}k_2^{\lambda} + \eta^{\nu\tau}\eta^{\rho\sigma}k_1^{\lambda}k_2^{\mu} + \eta^{\nu\sigma}\eta^{\rho\tau}k_1^{\lambda}k_2^{\mu} + \eta^{\lambda\tau}\eta^{\nu\rho}k_1^{\sigma}k_2^{\mu} - \\
&\eta^{\lambda\rho}\eta^{\nu\tau}k_1^{\sigma}k_2^{\mu} + \eta^{\lambda\nu}\eta^{\rho\tau}k_1^{\sigma}k_2^{\mu} + \eta^{\lambda\sigma}\eta^{\nu\rho}k_1^{\tau}k_2^{\mu} - \eta^{\lambda\rho}\eta^{\nu\sigma}k_1^{\tau}k_2^{\mu} + \eta^{\lambda\nu}\eta^{\rho\sigma}k_1^{\tau}k_2^{\mu} + \\
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&k_2 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_2 + \\
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\end{aligned}
$$

Gravity vs Gauge Feynman Diagrams

- Oh-shell three-point amplitude $(- - +)$:

$$A^{3,0}_{\rm graviton} = 4[(\varepsilon_1 \cdot \varepsilon_3)(p \cdot \varepsilon_2) - (\varepsilon_2 \cdot \varepsilon_3)(q \cdot \varepsilon_1)]^2 = i2[A^{3,0}_{\rm gluon}]^2$$

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- Vast simplification $\longrightarrow$ hidden structure

$$\text{Gravity} = \text{Gauge} \times \text{Gauge}$$

BCJ Double Copy Prescription

- Given CK dual amplitude of pure Yang-Mills $S_{\mathrm{YM}} = \frac{1}{2g^2}\int \mathrm{tr} F \wedge \star F$

$$A^{n,L}_{\mathrm{YM}} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

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- Gives an amplitude of $\mathcal{N} = 0$ supergravity

$$S_{\mathcal{N}=0} = \frac{1}{2\kappa^2}\int \star R - \frac{1}{d-2} d\varphi \wedge \star d\varphi - \frac{1}{2} \mathrm{e}^{-\frac{4}{d-2}\varphi} dB \wedge \star dB$$

where B is the Kalb-Ramond 2-form, φ is the dilaton

[Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

Implications and Applications

Computationally powerful: facilitates previously intractable calculations

- Miraculous cancellations: perturbatively finite quantum field theory of gravity?

- Black holes collisions and gravity wave astronomy: pushing the precision frontier

Implications and Applications

Computationally powerful: facilitates previously intractable calculations

- Miraculous cancellations: perturbatively finite quantum field theory of gravity?
- Black holes collisions and gravity wave astronomy: pushing the precision frontier

Conceptually provocative: is gravity really the square of gauge theory?

- Does CK duality and the double copy actually hold?
- What *is* CK duality?
- Can it be taken beyond amplitudes?

$\longrightarrow$ Lift CK duality and double copy to field theory?

Phase 2: Colour–Kinematic Duality Redux

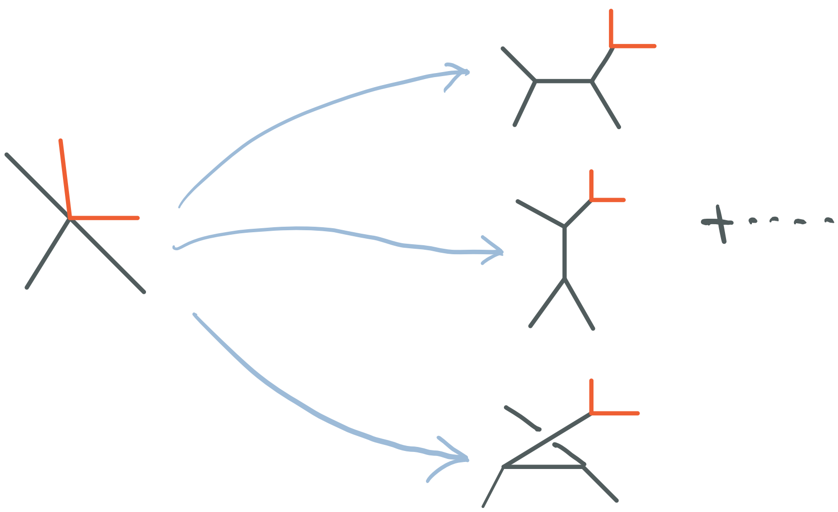
Colour–Kinematic Duality Redux

Manifest physical tree-level CK duality

- There is a YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

$$A\Box A + \partial AAA + \frac{\Box}{\Box}AAAA + \frac{\partial^3}{\Box^2}AAAAA + \cdots$$

Identically zero:
Colour Jacobi



[Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13]

Colour–Kinematic Duality Redux

Manifest physical tree-level CK duality

- This can be "strictified" to have only cubic interactions through infinite tower of auxiliaries [Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13; BJKMSW '21]

4-point vertex auxiliary

$$
\begin{aligned}
S^{\mathrm{YM}}_{\text{on-shell CK}} = \operatorname{tr}\int d^D x\, & \tfrac{1}{2}A_\mu \Box A^\mu + \tfrac{1}{2} g \partial_\mu A_\nu [A^\mu, A^\nu] \\
& + \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\
& + C^{\mu\nu} \Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \Box \bar{C}_{\mu\nu\kappa\lambda} + \\
& + g C^{\mu\nu}[A_\mu, A_\nu] + g\partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2}\partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu} A_{\kappa]}, A_\lambda] \\
& + g \bar{C}^{\mu\nu}\left(\tfrac{1}{2}[\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]\right) + \cdots
\end{aligned}
$$

Identically zero:
Colour Jacobi

5-point vertex auxiliary

- Purely cubic Feynman diagrams:

$$
A^{n,0}_{\mathrm{YM}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0
$$

Colour–Kinematic Duality Redux

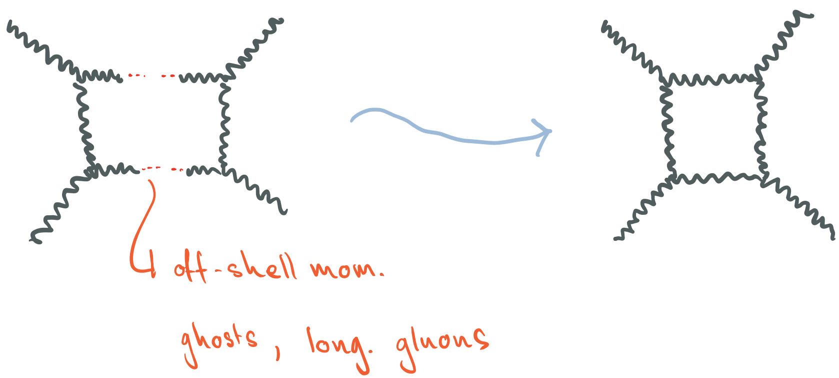
The need for unphysical BRST CK duality

- Include off-shell unphysical/ghost modes in the external states, the full BRST-extended state space

$$({A_\mu}^a, b^a, c^a, \bar{c}^a)$$

- BRST CK duality ensures consistent double copy of the BRST charge Q

- Intuition: unphysical off-shell modes propagate in the loops of Feynman diagrams



Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

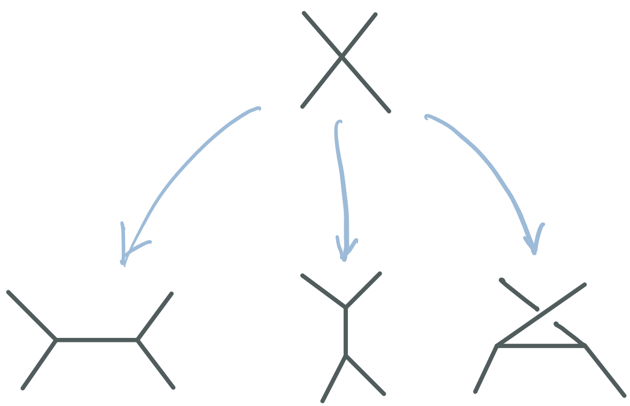
- Relax transversality $p_i \cdot \varepsilon_i \neq 0$ for external states $\Rightarrow$ CK duality fails

$$A^{n,0}_{\mathrm{YM}} \quad \rightarrow \quad \hat{A}^{n,0}_{\mathrm{YM}}$$

Physical amplitude

BRST-extended 'amplitude'

- By analogy with physical gluons can compensate with new vertices [BJKMSW '20]:



- Non-zero: BRST-extended amplitude changes $\hat{A}^{n,0}_{\mathrm{YM}} \rightarrow \hat{A}'^{n,0}_{\mathrm{YM}}$ (but physical amplitude $A^{n,0}_{\mathrm{YM}}$ is invariant)

Colour-Kinematic Duality Redux

Tree-level CK duality for on-shell longitudinal gluons

- New vertices are necessarily of the form

$$(\partial \cdot A) Y[A]$$

- Can add through the gauge-fixing fermion $\Psi' = \Psi - 2\xi \bar{c} Y$

$$\begin{array}{rccccc} \text{Gauge-fixing } G[A]: & \partial \cdot A & \mapsto & G'[A] & = & \partial \cdot A - 2\xi Y \\ \text{Nakanishi-Lautrup } b: & b & \mapsto & b' & = & b + Y \end{array}$$

$$S^{\mathrm{YM}}_{\mathrm{BRST}} \mapsto S^{\mathrm{YM}}_{\mathrm{BRST}} + \int (\partial \cdot A) Y + \cdots$$

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$$S^{\text{YM}}_{\text{BRST}} \mapsto S^{\text{YM}}_{\text{BRST}} + \int (\partial \cdot A)Y + \cdots$$

- Longitudinal CK duality $\Leftrightarrow$ gauge choice [BJKMSW '20, '21]

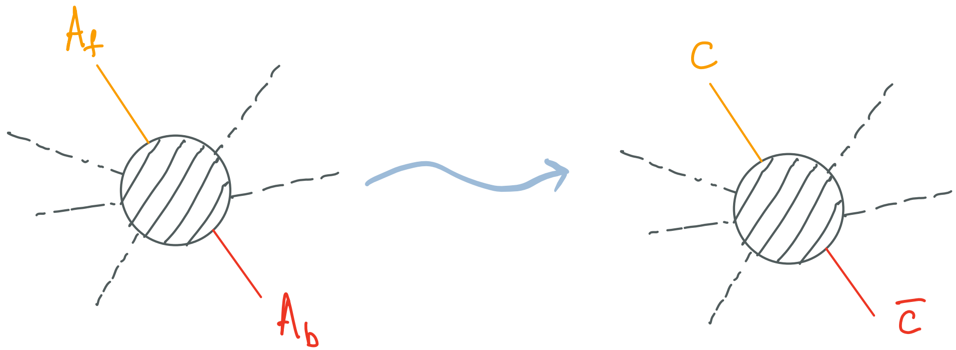
Colour-Kinematic Duality Redux

Tree-level CK duality for on-shell ghosts

- Use on-mass-shell BRST Ward identities

$$0 = \langle 0|[Q^{\rm lin}_{\rm YM}, O_1 \cdots O_n]|0\rangle$$

$$Q^{\rm lin}_{\rm YM} A_{\sf phys} = 0, \quad Q^{\rm lin}_{\rm YM} A_{\sf f} = c, \quad Q^{\rm lin}_{\rm YM} b = \bar{c}$$



- On-mass-shell BRST Ward identities transfers CK duality onto ghosts through

$$S^{\sf ghost}_{\sf YM} = \int d^D x \; {\rm tr} \bar{c} Q(\partial^\mu A_\mu - 2\xi Y)$$

Colour-Kinematic Duality Redux

On-shell tree-level CK manifesting BRST action

- Introduce further auxiliary gluons and ghosts [BJKMSW '20, '21]:

Auxiliary gluon

$$
\begin{aligned}
\mathcal{L}^{\text{YM}}_{\text{BRST CK}} \;=\;& \tfrac{1}{2} A_{a\mu} \Box A^{\mu a} - \bar{c}_a \Box c^a + \tfrac{1}{2} b_a \Box b^a + \xi\, b_a \sqrt{\Box}\, \partial_\mu A^{\mu a} \\
& - K^{\mu}_{1a} \Box \bar{K}^{1a}_{\mu} - K^{\mu}_{2a} \Box \bar{K}^{2a}_{\mu} - g f_{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) \\
& - \tfrac{1}{2} B^{\mu\nu\kappa}_a \Box B^a_{\mu\nu\kappa} + g f_{abc} \left(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B^a_{\kappa\mu\nu} \right) A^{\mu b} A^{\nu c} \\
& - g f_{abc} \left\{ K^{a\mu}_{1} (\partial^\nu A^b_\mu) A^c_\nu + [(\partial^\kappa A^a_\kappa) A^{b\mu} + \bar{c}^a \partial^\mu c^b] \bar{K}^{1c}_\mu \right\} \\
& + g f_{abc} \left\{ K^{a\mu}_{2} \left[(\partial^\nu \partial_\mu c^b) A^c_\nu + (\partial^\nu A^b_\mu) \partial_\nu c^c \right] + \bar{c}^a A^{b\mu} \bar{K}^{2c}_\mu \right\} + \cdots
\end{aligned}
$$

Auxiliary ghost

- Cubic Feynman diagrams yield CK dual tree amplitudes for physical gluons and unphysical longitudinal modes, ghosts and auxiliary fields (on-shell)

Colour–Kinematic Duality Redux

Lifting to off-shell CK duality

- Off-shell momenta $p^2 \neq 0$: resulting CK duality violations are compensated by vertices $f\Box\phi$ generated by generically non-local field redefinitions:

$$\phi \mapsto \phi + f(\phi), \qquad \phi\Box\phi \mapsto \phi\Box\phi + f\Box\phi + \cdots$$

- Strictify again: off-shell tree-level BRST CK duality is rendered manifest

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Price to pay

- Jacobian determinants $\rightarrow$ counterterms ensuring unitarity

$$\det\left(\mathbb{1} + \frac{\delta f(\phi)}{\delta\phi}\right) = \int \mathcal{D}\bar{\chi}\,\mathcal{D}\chi\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int\left(\bar{\chi}_I\chi^I + \bar{\chi}_I\frac{\delta f^I}{\delta\phi^J}\chi^J\right)}$$

- No reason to think such terms will preserve CK duality: in this sense, our off-shell CK duality is anomalous on the physical Hilbert space
- Two-loop CK duality with cubic Feynman rule compatible numerators is impossible [Bern, Davies, Nohle '15]
- Here, we understand this impossibility as a CK duality anomaly

Summary: manifest off-shell BRST-Lagrangian CK duality

- YM BRST-action with manifest off-shell CK duality

$$S^{\text{YM}}_{\text{BRST CK}} = \int C_{ij}c_{ab}A^{ia}\Box A^{ja} + F_{ijk}f_{abc}A^{ia}A^{jb}A^{kc}$$

- Rendered cubic with infinite tower of aux. fields

$$A^{ia} = (A_\mu{}^a, b^a, \bar{c}^a, c^a, \underbrace{B_{\mu\nu\rho}{}^a, \bar{K}_\mu{}^a, \dots}_{\text{auxiliaries}})$$

- c_{ab}, f^{abc} colour metric and structure constants

- C_{ij}, F^{ijk} kinematic metric and structure constants: differential operators that satisfy the same identities as c_{ab}, f^{abc} as operator equations

$$\begin{array}{cccc} c_{ab} = c_{(ab)} & f_{abc} = f_{[abc]} & c_{a(b}f^a_{c)d} = 0 & f_{[ab|d}f^d_{c]e} = 0 \\ C_{ij} = C_{(ij)} & F_{ijk} = F_{[ijk]} & C_{i(j}F^i_{k)l} = 0 & F_{[ij|l}F^l_{|k]m} = 0 \end{array}$$

CK duality is rendered a symmetry of the BRST action

§3.

Phase 3: Syngamy

BRST Lagrangian Syngamy

Syngamic reproduction of factorable theories

Parent theories

$$c_{IJ}\phi^I \Box \phi^J + f_{IJK}\phi^I\phi^J\phi^K$$

$$\tilde{c}_{\tilde{I}\tilde{J}}\tilde{\phi}^{\tilde{I}} \Box \tilde{\phi}^{\tilde{J}} + \tilde{f}_{\tilde{I}\tilde{J}\tilde{K}}\tilde{\phi}^{\tilde{I}}\tilde{\phi}^{\tilde{J}}\tilde{\phi}^{\tilde{K}}$$

Factorisation

$$C_{ij}c_{ab}\Phi^{ia}\Box\Phi^{jb} + F_{ijk}f_{abc}\Phi^{ia}\Phi^{jb}\Phi^{kc}$$

$$\tilde{c}_{\tilde{a}\tilde{b}}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{\tilde{a}\tilde{\imath}}\Box\tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\tilde{\Phi}^{\tilde{a}\tilde{\imath}}\tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}}$$

Daughter theories

$$C_{ij}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{i\tilde{\imath}}\Box\Phi^{j\tilde{\jmath}} + F_{ijk}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{i\tilde{\imath}}\Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$$

$$\tilde{c}_{\tilde{a}\tilde{b}}C_{ij}\Phi^{\tilde{a}i}\Box\Phi^{\tilde{a}j} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}F_{ijk}\Phi^{\tilde{a}i}\Phi^{\tilde{b}j}\Phi^{\tilde{c}k}$$

$$c_{ab}\tilde{C}_{\tilde{\imath}\tilde{\jmath}}\Phi^{a\tilde{\imath}}\Box\Phi^{a\tilde{\jmath}} + f_{abc}\tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\Phi^{a\tilde{\imath}}\Phi^{b\tilde{\jmath}}\Phi^{c\tilde{k}}$$

$$c_{ab}\tilde{c}_{\tilde{a}\tilde{b}}\Phi^{a\tilde{a}}\Box\Phi^{a\tilde{b}} + f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}}$$

BRST Double Copy of Yang–Mills

- Double copy of Yang–Mills BRST-action:

$$A^{ia} = (A_\mu{}^a, \dots) \qquad S^{\mathrm{YM}}_{\mathrm{CK}} = C_{ij} c_{ab} A^{ia} \Box A^{ja} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

Kalb–Ramond 2-form

$$A^{i\tilde{\imath}} = (h_{\mu\nu}, B_{\mu\nu}, \varphi, \dots) \qquad S^{\mathcal{N}=0}_{\mathrm{DC}} = C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

Graviton Dilaton

$$\begin{aligned}
\mathcal{L}^{\mathcal{N}=0}_{\mathrm{DC}} = {} & \tfrac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + \tfrac{1}{2} \varpi_\mu \Box \varpi^\mu + \xi^2 (\partial^\mu \varpi_\mu)^2 + \tfrac{1}{2} \pi \Box \pi \\
& - 2\xi \varpi^\nu \Box^{\frac{1}{2}} \partial^\mu h_{\mu\nu} - 2\xi\pi \Box^{\frac{1}{2}} \partial_\mu \varpi^\mu + 2\xi^2 \pi \partial_\mu \partial_\nu h^{\mu\nu} \\
& - 2\bar{X}_\mu \Box X^\mu - \delta \Box \delta - 2\bar{\beta} \Box \beta - 2\xi\beta \Box^{\frac{1}{2}} \partial_\mu \bar{X}^\mu + 2\xi\bar{\beta} \Box^{\frac{1}{2}} \partial_\mu X^\mu \\
& + \tfrac{1}{2} B_{\mu\nu} \Box B^{\mu\nu} - 2\bar{\Lambda}_\mu \Box \Lambda^\mu + \alpha_\mu \Box \alpha^\mu + \xi^2 (\partial^\mu \alpha_\mu)^2 + \varepsilon \Box \varepsilon - \bar{\lambda} \Box \lambda - 2\bar{\gamma} \Box \gamma \\
& - 2\xi\alpha^\nu \Box^{\frac{1}{2}} \partial^\mu B_{\mu\nu} - 2\xi\gamma \Box^{\frac{1}{2}} \partial_\mu \bar{\Lambda}^\mu + 2\xi\bar{\gamma} \Box^{\frac{1}{2}} \partial_\mu \Lambda^\mu + \cdots
\end{aligned}$$

Graviton-dilaton

KR 2-form

- Canonical field redefinition to Fierz-Pauli + Kalb-Ramond + dilaton action

$$QB = d\Lambda, \qquad Q\Lambda = d\lambda, \qquad Q\lambda = 0$$

- Cubic Einstein-Hilbert BRST action explicitly recovered [LB, Nagy '20]

BRST-Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- How do we know $S^{\mathcal{N}=0}_{\mathrm{DC}}$ is equivalent to true $S^{\mathcal{N}=0}_{\mathrm{BRST}}$?

BRST-Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- How do we know $S^{\mathcal{N}=0}_{\text{DC}}$ is equivalent to true $S^{\mathcal{N}=0}_{\text{BRST}}$?

- Semi-classical equivalence (requires physical tree-level CK duality)

$$
\begin{tikzcd}
F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \arrow[r] \arrow[d, "\text{Feynman rules}"'] & F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}} \arrow[d, "\text{Feynman rules}"] \\
\sum \frac{nc}{d} \arrow[r] \arrow[ur, phantom, "\text{Double Copy}"] & \sum \frac{n\tilde{n}}{d}
\end{tikzcd}
$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity

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$$\begin{tikzcd}
F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \arrow[rr] \arrow[d, bend right, "\text{Feynman rules}"'] & & F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}} \arrow[d, bend left, "\text{Feynman rules}"] \\
\sum \frac{nc}{d} \arrow[rr] \arrow[urr, phantom, "\text{Double Copy}"] & & \sum \frac{n\tilde{n}}{d}
\end{tikzcd}$$

- $\Rightarrow$ physical (h, B, φ) tree-level amplitudes of $\mathcal{N} = 0$ supergravity
- Cf. [Bern, Dennen, Huang, Kiermaier '10] for gravitons up to 6 points
- Quantum consistency: is there some Q such that

$$QS^{\mathcal{N}=0}_{\text{DC}} = 0, \qquad Q^2 = 0$$

BRST-Lagrangian Syngamy

BRST-Lagrangian CK duality $\Rightarrow$ consistent syngamy

- How do we know $S_{\text{DC}}^{\mathcal{N}=0}$ is equivalent to true $S_{\text{BRST}}^{\mathcal{N}=0}$?
- Semi-classical equivalence (requires physical tree-level CK duality)

$$\begin{tikzcd}
F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc} \arrow[r, "\text{Double Copy}"] \arrow[d, bend right, "\text{Feynman rules}"'] & F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}} \arrow[d, bend left, "\text{Feynman rules}"] \\
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- Quantum consistency: is there some Q such that

$$QS_{\text{DC}}^{\mathcal{N}=0} = 0, \qquad Q^2 = 0$$

Answer: double-copy operator Q_{DC} (requires off-shell BRST CK duality)

BRST-Lagrangian Syngamy

Double copy of BRST charge

- Double copy of BRST-action implies double copy BRST operator Q_{DC}

$$QA^{ia} = Q^{i}{}_{j}A^{ja} + Q^{i}{}_{jk}f^{a}{}_{bc}A^{jb}A^{kc} \qquad \tilde{Q}\tilde{A}^{\tilde{a}\tilde{\imath}} = Q^{\tilde{\imath}}{}_{\tilde{\jmath}}\tilde{A}^{\tilde{b}\tilde{\jmath}} + \tilde{f}^{\tilde{a}}{}_{\tilde{b}\tilde{c}}\tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}\tilde{A}^{\tilde{b}\tilde{\jmath}}\tilde{A}^{\tilde{c}\tilde{k}}$$

$$Q_{\mathrm{DC}} = \underbrace{Q^{i}{}_{j}A^{j\tilde{\imath}} + Q^{i}{}_{jk}F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}A^{j\tilde{\jmath}}A^{k\tilde{k}}}_{Q_L} + \underbrace{Q^{\tilde{\imath}}{}_{\tilde{\jmath}}A^{i\tilde{\jmath}} + F^{i}{}_{jk}Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}}A^{j\tilde{\jmath}}A^{k\tilde{k}}}_{Q_R}$$

- Since F^{ijk} satisfy the same identities as f^{abc} and $QS^{\mathrm{YM}}_{\mathrm{BRST}} = 0$, $Q^2 = 0$ can only rely on generic properties of f^{abc}:

$$Q_{\mathrm{DC}}S_{\mathrm{DC}} = 0, \qquad Q^2_{\mathrm{DC}} = 0$$

- Semi-classical equivalence + $Q_{\mathrm{DC}} \Rightarrow$ quantum equivalence

BRST-Lagrangian Syngamy

Double Copy Symmetries

- Yang-Mills gauge $\Rightarrow$ diffeomorphisms and 2-form gauge symmetries:

$$Q_{\text{DC}} = Q_{\text{diffeo}} + Q_{\text{2-form}} + \text{trivial symmetries}$$

$$Q_{\text{2-form}} B = \Lambda, \quad Q_{\text{2-form}} \Lambda = \lambda \quad Q_{\text{2-form}} \lambda = 0$$

- Double copy of symmetries generalises, e.g.

$$\text{global susy} \quad \times \quad \text{gauge} \quad \rightarrow \quad \text{local susy}$$

See also [Anastasiou, LB, Duff, Hughes, Nagy '14]

Super Yang–Mills and Supergravity

BRST-Lagrangian double copy

- Irreducible super Yang–Mills multiplets are CK duality respecting: same game
  Cf. for example **[Bjerrum–Bohr, Damgaard, Vanhove '09]**

- (Type I super Yang–Mills)^2 = Type IIA/B supergravity

$$A^{Ia} = (A^{ia}, \Psi^{xa}) = (A_\mu{}^a, \psi_\alpha{}^a, \text{ghosts}, \text{aux})$$

Gluino

$$A^{J\tilde{J}} = (h_{\mu\nu}, B_{\mu\nu}, \phi, \underbrace{\Psi_{\alpha\nu}, \Psi_{\mu\beta}}_{\text{Gravitini}}, F_{\alpha\beta}, \text{ghosts}, \text{aux})$$

RR field strengths

- Local NS-R sector susy follows from super Yang–Mills factors

$$\mathcal{Q}_\alpha A_\mu{}^a = \delta^a{}_b \gamma_{\mu\alpha}{}^\beta \psi_\beta{}^b + \cdots \quad \longrightarrow \quad \mathcal{Q}_\alpha h_{\mu\nu} = \gamma_{(\mu\alpha}{}^\beta \Psi_{\beta\nu)} + \cdots$$

- Super $\eta, \bar{\eta}$ and Nielsen, Kallosh χ ghosts

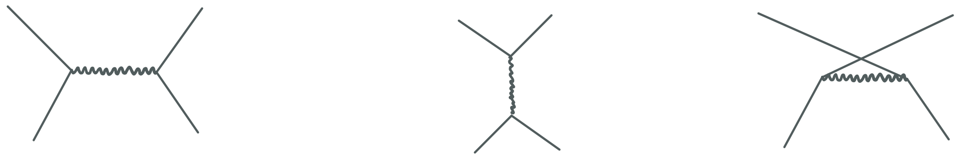
$$\bar{c} \otimes \psi \sim \bar{\eta}\,, \qquad c \otimes \psi \sim \eta\,, \qquad b \otimes \psi \sim \chi$$

Cf. **[Anastasiou, LB, Duff, Hughes, Nagy '14]**

Super Yang–Mills and Supergravity

CK duality: the mother of all symmetries

- CK duality $\Rightarrow$ supersymmetry [Chiodaroli, Jin, Roiban '13]



$$\frac{i}{2}\left(\frac{(\bar{u}_1\gamma_\mu v_2)(\bar{u}_3\gamma^\mu v_4)c_s}{s}+\frac{(\bar{u}_2\gamma_\mu v_3)(\bar{u}_1\gamma^\mu v_4)c_s}{t}+\frac{(\bar{u}_3\gamma_\mu v_1)(\bar{u}_2\gamma^\mu v_4)c_s}{u}\right)$$

- CK duality requires Fierz identity: $\mathcal{N}=1$ super Yang–Mills in $D=3,4,6,10$

$$(\bar{u}_1\gamma_\mu v_2)(\bar{u}_3\gamma^\mu v_4)+(\bar{u}_2\gamma_\mu v_3)(\bar{u}_1\gamma^\mu v_4)+(\bar{u}_3\gamma_\mu v_1)(\bar{u}_2\gamma^\mu v_4)=0$$

- Mother of all symmetries: off-shell CK duality implies supersymmetry directly

$$S^{\mathrm{SYM}}_{\mathrm{BRST}}=C_{IJ}c_{ab}A^{Ia}\Box A^{Ja}+F_{IJK}f_{abc}A^{Ia}A^{Jb}A^{Kc}$$

$$\delta_\epsilon A^{ia}=F^{i}{}_{xy}\Psi^{xa}\epsilon^{y},\quad \delta_\epsilon\Psi^{xa}=F^{x}{}_{jy}A^{ja}\epsilon^{y}$$

Super Yang–Mills and Supergravity

Ramond-Ramond sector

- Double copy $\psi_\alpha \otimes \psi_\beta$ gives field strengths $F_{\alpha\beta}$, not potentials [Nagy '14]:
  - Representation theory
$$\begin{aligned} \text{IIA:} \quad \overline{16} \otimes 16 &= 1 \oplus 45 \oplus 210 \\ \text{IIB:} \quad 16 \otimes 16 &= 10 \oplus 120 \oplus 126 \end{aligned}$$
  - The BRST transformation of the gluino has no linear contribution, $Q_{\mathbf{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths

Super Yang–Mills and Supergravity

Ramond-Ramond sector

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  - The BRST transformation of the gluino has no linear contribution, $Q_{\mathbf{BRST}}\psi = [c, \psi]$, so $\psi \otimes \psi$ cannot transform as a potential
  - R-R background fields couple to worldsheet through field strengths
- Type IIA/B action can be written in terms of field strengths alone for RR sector:

$$F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star F_4 + B_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + B_2 \wedge B_2 \wedge F_2 \wedge \tilde{F}_4 - \tfrac{1}{3} B_2 \wedge B_2 \wedge B_2 \wedge F_2 \wedge F_2$$

Sen's mechanism from double copy Ramond-Ramond sector

- Double copy RR field strengths are elementary fields:

$$
\begin{aligned}
\mathcal{L}^{\mathsf{RR}}_{\mathsf{DC}} &= \overline{F}^{\alpha\beta}\,\frac{1}{\Box}\,\partial\!\!\!/_{\alpha}{}^{\alpha'}\,\partial\!\!\!/_{\beta}{}^{\beta'}\,F_{\alpha'\beta'}+\cdots \\
&\to -\tfrac{1}{2}\left(F\wedge\star F-\mathrm{d}F\wedge\star\,\Box^{-1}\mathrm{d}F\right)+\cdots \\
&\to -\tfrac{1}{2}F\wedge\star F-\xi B\wedge\mathrm{d}F-\tfrac{1}{2}B\wedge\star\,\Box B+\cdots \\
&\to -\tfrac{1}{2}F\wedge\star F-\xi B\wedge\mathrm{d}F+\tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B
\end{aligned}
$$

Super Yang–Mills and Supergravity

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&\to -\tfrac{1}{2}\left(F\wedge\star F - \mathrm{d}F\wedge\star\Box^{-1}\mathrm{d}F\right) + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F - \tfrac{1}{2}B\wedge\star\Box B + \cdots \\
&\to -\tfrac{1}{2}F\wedge\star F - \xi B\wedge\mathrm{d}F + \tfrac{1}{2}\mathrm{d}B\wedge\star\mathrm{d}B
\end{aligned}
$$

- Sen's mechanism [Sen '15] generalized to arbitrary (as opposed to self-dual) field strengths [BJKMSW '21]

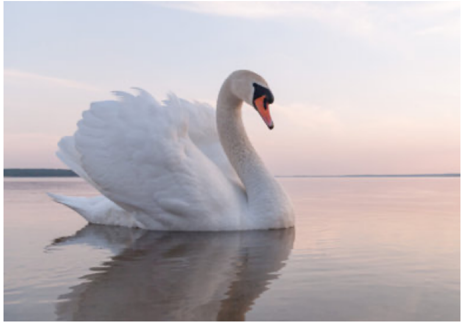
- Sen's mechanism was motivated by IIB string field theory, where the RR sector is naturally given in terms of bispinors - natural double copy shadow

Phase 4: Closing Remarks

The Homotopy Algebra of Colour-Kinematics Duality

- CK duality is a symmetry of action: kinematic algebra?
  Hands on quantum field theory

The Homotopy Algebra of Colour-Kinematics Duality



- CK duality is a symmetry of action: kinematic algebra?
  Hands on quantum field theory

- $BV_\infty^\square$ homotopy algebra
  Abstract nonsense [BJKMSW '22 (to appear)]
  See Tommaso's talk up next

Michel Reiterer [1912.03110]

Gluons, quarks...
Scattering amplitudes
Symmetries of Nature

$BV_\infty^\square$-algebras —Homotopy transfer— Exact CK duality

$BV_\infty^\square$-operad —Koszul resolution— $BV^\square$-operad

Cat(Hodge)

Symmetric monoidal category

Twisted tensor product
of (co)algebras

Homotopy algebras
Koszul duality
Operads
Symmetric monoidal categories

$H^\square$ ——— Hodge Complexes

Hopf algebras with central element $\square$

Natural generalisation of familiar Hodge–de Rham complex

- CK duality: a symmetry of Nature as a mug is a donut!

Where next?

- What is the relationship between the kinematic $BV^{\square}$-algebras and the other kinematic algebra structures in the literature?

- Can it be leveraged for computational efficiency?

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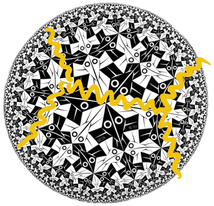
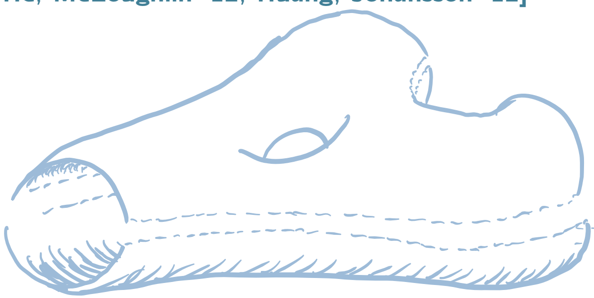
- What is the relationship between the kinematic $BV^{\square}$-algebras and the other kinematic algebra structures in the literature?



- Can it be leveraged for computational efficiency?
- AdS space [Farrow, Lipstein, McFadden '19; Zhou '21; Diwakar, Herderschee, Roiban, Teng '21 ...]
  $\rightarrow$ Hopf algebra of universal enveloping algebra of AdS isometries
- Bagger-Lambert-Gustavsson [Bargheer, He, McLoughlin '12; Huang, Johansson '12]
  $\rightarrow$ m-ary $BV^{\square}$ operads



- Matter coupling [Johansson, Ochirov '14]
  $\rightarrow$ many-sorted $BV^{\square}$ operads
- String theory

$$\{d, h\} = \square \quad \longrightarrow \quad \{Q, b_0\} = L_0$$

Cf. BV_∞ structure on TVOA [Galvez–Carrillo, Tonks, Vallette '09] lifting the BV-algebra structure on the BRST (co)homology [Lian-Zuckerman '93]

Thanks for listening